

Canonical superenergy and angular supermomentum complexes in general relativity and some of their applications

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Many years ago we have introduced into general relativity, **GR**, the canonical superenergy tensors, S_i^k , and the canonical angular supermomentum tensors, $S^{ikl} = (-)S^{kil}$, matter and gravitation. We have obtained these tensors by special averaging of the differences of the canonical energy-momentum and canonical angular momentum. The averaging was performed in Riemann normal coordinates, **RNC(P)**; **P** is beginning of these coordinates.

About four years ago we have observed that these tensors can also be obtained in other, simpler way, by using the canonical superenergy and angular super momentum complexes, ${}_K S_i^k$, and, ${}_K S^{ikl} = (-){}_K S^{kil}$, respectively. Such complexes can be introduced into **GR** in a natural way starting from canonical energy-momentum and angular momentum complexes.

In this paper, at first, we define the canonical superenergy and angular supermomentum complexes in **GR** and then, we apply them to analyze of a closed system, **CS**, Trautman's radiative spacetimes, **TRS**, and Friedman universes, **FU**.

Finally, we compare these complexes and the results obtained with their help with the canonical superenergy and angular supermomentum tensors and results obtained with them in past.

In Appendix, for convenience, we summarize our old approach to canonical superenergy and angular supermomentum tensors.

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I. THE CANONICAL SUPERENERGY AND ANGULAR SUPERMOMENTUM COMPLEXES IN GR

We begin with general remark that in the whole paper we will use the same signature $(+---)$ and notation as used in the last edition of the famous book [1]. Latin indices take values 0, 1, 2, 3 and Greek indices range values 1, 2, 3.

In the framework of general relativity, **GR**, as a consequence of the Einstein Equivalence Principle, **EEP**, the gravitational field *has non-tensorial strengths* $\Gamma_{kl}^i = \{i_{kl}\}$ and *admits no energy-momentum tensor*. By using standard procedures of the classical field theory one can only attribute to this field *gravitational energy-momentum pseudotensors*. The leading object of such a kind is the *canonical gravitational energy-momentum pseudotensor*, ${}_E t_i^k$, proposed already in past by Einstein. This pseudotensor is a part of the *canonical energy-momentum complex*, ${}_E K_i^k$, in **GR**.

The canonical complex, ${}_E K_i^k$, firstly obtained by using standard field-theoretic procedure to general relativistic Lagrangian (See, e.g., [1–3]), can be most easily obtained by rewriting Einstein equations to the superpotential form

$${}_E K_i^k := \sqrt{|g|} (T_i^k + {}_E t_i^k) = {}_F U_i^{[kl]}{}_{,l}, \quad (1)$$

where $T^{ik} = T^{ki}$ is the symmetric energy-momentum tensor for matter, $g = \det[g_{ik}]$, and

$$\begin{aligned} {}_E t_i^k &= \frac{c^4}{16\pi G} \{ \delta_i^k g^{ms} (\Gamma_{mr}^l \Gamma_{sl}^r - \Gamma_{ms}^r \Gamma_{rl}^l) \\ &+ g^{ms}{}_{,i} [\Gamma_{ms}^k - \frac{1}{2} (\Gamma_{tp}^k g^{tp} - \Gamma_{tl}^l g^{kt})] g_{ms} \\ &- \frac{1}{2} (\delta_s^k \Gamma_{ml}^l + \delta_m^k \Gamma_{sl}^l) \}; \end{aligned} \quad (2)$$

$${}_F U_i^{[kl]} = \frac{c^4}{16\pi G} \frac{g_{ia}}{\sqrt{|g|}} [(-g)(g^{ka} g^{lb} - g^{la} g^{kb})]_{,b}. \quad (3)$$

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${}_E t_i^k$ are components of the canonical energy-momentum pseudotensor for gravitational field, and ${}_F U_i^{[kl]} = (-) {}_F U_i^{[lk]}$ are Freud's superpotential.

$${}_E K_i^k := \sqrt{|g|} (T_i^k + {}_E t_i^k) \quad (4)$$

are components of the *Einstein canonical energy-momentum complex for matter and gravity* in **GR**.

In consequence of (1) the complex ${}_E K_i^k$ satisfies the *continuity equations* (=local conservation laws)

$${}_E K_i^k{}_{,k} = {}_F U_i^{[kl]}{}_{,lk} = 0. \quad (5)$$

In very special cases, e.g., in the case of a closed system, **CS**, [4] one can obtain from these continuity equations the four reasonable integral conservation laws energy and momentum.

From (5) one can also obtain, by successive differentiation, infinitely many other continuity equations

$$[\sqrt{|g|} (T_i^k + {}_E t_i^k)]_{,klm\dots rst} = {}_F U_i^{[kl]}{}_{,klm\dots rst} = 0, \quad (6)$$

especially, one can get from (5) the following continuity equations

$$[\delta^{ab} \sqrt{|g|} (T_i^k + {}_E t_i^k)]_{,kab} = {}_F U_i^{[kl]}{}_{,lkab} \delta^{ab} = 0, \quad (7)$$

or, by using Schwarz's Lemma, the equations

$$\{[\delta^{ab} \sqrt{|g|} (T_i^k + {}_E t_i^k)]_{,ab}\}_{,k} = [(\delta^{ab} {}_F U_i^{[kl]})_{,lab}]_{,k} = 0. \quad (8)$$

After introducing differential operator

$$\Delta^{(4)} := \partial_o^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 \equiv \delta^{ab} \partial_a \partial_b \quad (9)$$

one can write the equations (8) in the form

$$(\Delta^{(4)} {}_E K_i^k)_{,k} = (\Delta^{(4)} {}_F U_i^{[kl]})_{,l} = 0. \quad (10)$$

The continuity equations (6), which follow from (5), *have, in general, no new physical meaning* except the case (7)-(8) in which analytic 4-Laplacian $\Delta^{(4)} = \delta^{ab} \partial_a \partial_b$ occurs.

Namely, from the constructive definition of the canonical superenergy tensors, ${}_g S_i^k$, ${}_m S_i^k$, gravity and matter, given in [5, 6] (See also Appendix), one can easily see that these tensors are exactly 4-dimensional analytic Laplacians

$$\Delta^{(4)} T_i^k(P) = \delta^{ab} T_i^k{}_{,ab}(P), \quad \Delta^{(4)} {}_E t_i^k(P) = \delta^{ab} {}_E t_i^k{}_{,ab}(P), \quad (11)$$

calculated in Riemann normal coordinates, **RNC(P)**, taken at the point **P** [**P** = beginning of the **RNC(P)**], and then, expressed covariantly in terms of normal tensors, tensor extensions [7, 8], and 4-velocity of a fiducial observer **O**, which is at rest at point **P**.

Therefore, the quantity on the left hand side of the continuity equations (8), i.e.,

$$\Delta_E^{(4)} K_i^k = [\delta^{ab} \sqrt{|g|} (T_i^k + {}_E t_i^k)]_{,ab}, \quad (12)$$

when taken at beginning **P** of the **RNC(P)** and covariantly expressed, gives us exactly the total canonical superenergy tensor, matter and gravity, ${}_m S_i^k(P; v^l) + {}_g S_i^k(P; v^l)$ (See [5, 6] and Appendix).

From these reasons we call the analytic Laplacian

$$(\delta^{ab} {}_E K_i^k)_{,ab} = (\delta^{ab} {}_F U_i^{[kl]})_{,l}{}_{,ab} \quad (13)$$

the *canonical superenergy complex*, matter and gravitation, and denote it ${}_K S_i^k$. [15]

From (8) we see that the complex, ${}_K S_i^k$, like the canonical energy-momentum complex, ${}_E K_i^k$, satisfies continuity equations

$${}_K S_i^k{}_{,k} = 0. \quad (14)$$

The components of the complex, ${}_K S_i^k$, have the same dimensions as components of the canonical superenergy tensors, i.e., they have dimensions of the energy-momentum divided by m^2 : $[{}_K S_i^k] = \frac{[T_i^k]}{m^2}$.

We would like to emphasize that the canonical superenergy density, ϵ_s , calculated from superenergy complex or from superenergy tensors exactly corresponds to Appel's energy of acceleration which plays important role in classical mechanics (See, e.g., [5, 6, 10]).

Consequently, the canonical superenergy complex, ${}_K S_i^k$, and the continuity equations (14) surely can have a physical meaning (Like the canonical superenergy tensors, ${}_g S_i^k$, ${}_m S_i^k$, gravitation and matter).

In analogical way one can introduce into **GR** the canonical angular supermomentum complex, ${}_K S^{ijk} = (-) {}_K S^{jik}$, matter and gravitation.

Namely, we start from the canonical angular momentum complex, $M^{ijk} = (-) M^{jik}$, in **GR** [11, 12]

$$\begin{aligned} M^{ijk} &= (-) M^{jik} := x_{BT}^i K^{jk} - x_{BT}^j K^{ik} + {}_F U^{i[jk]} - {}_F U^{j[ik]} \\ &= (x_F^i U^{j[kl]} - x_F^j U^{i[kl]})_{,l} =: M^{[ij][kl]}_{,l}. \end{aligned} \quad (15)$$

Here

$${}_{BT} K^{ik} := g^{ij} {}_E K_j^k + {}_F U_j^{[kl]} g^{ij}_{,l} = \sqrt{\text{vert}g} (T^{ik} + {}_{BT} t^{ik}) \neq {}_{BT} K^{ki} \quad (16)$$

are components of the *Bergmann-Thomson energy-momentum complex* in **GR** [11, 12], ${}_F U^{i[jk]} := g^{il} {}_F U_l^{[jk]}$, and double antisymmetric quantity

$$M^{[ij][kl]} := x_F^i U^{j[kl]} - x_F^j U^{i[kl]} \quad (17)$$

is *angular momentum superpotential*.

${}_{BT} t^{ik} \neq {}_{BT} t^{ki}$ is the *Bergmann-Thomson* gravitational energy- momentum pseudotensor (See Appendix).

The non-tensorial (that's why "complex") complex, $M^{ijk} = (-) M^{jik}$, satisfies, in consequence of (15), the continuity equations

$$M^{ijk}_{,k} = M^{[ij][kl]}_{,lk} = 0. \quad (18)$$

From (18) one can get for a closed system the reasonable six integral conservation laws for angular momentum [4, 11, 12].

We introduce the *canonical angular supermomentum complex* in **GR** (also non-tensorial), ${}_K S^{ijk} = (-) {}_K S^{jik}$, by definition

$${}_K S^{ijk} := (\delta^{ab} M^{ijk})_{,ab} = \Delta^{(4)} M^{ijk}. \quad (19)$$

As consequence of the continuity equations (18) and Schwarz's Lemma one has

$${}_K S^{ijk}_{,k} = 0, \quad (20)$$

i.e., one has continuity equations for canonical angular supermomentum.

The quantity (19), when taken at the beginning **P** of the **RNC(P)** and expressed covariantly in terms of the normal tensors, tensors extensions, and 4-velocity of a fiducial observer **O** at rest in **P** exactly gives our total canonical angular supermomentum tensor

$$S^{ikl} = (-) S^{kil} = {}_g S^{ikl} + {}_m S^{ikl}, \quad (21)$$

gravitation and matter [5, 6] (See also Appendix).

Thus, the name *angular supermomentum complex* for quantity (19) is justified.

In the following we will apply the above introduced canonical superenergy and angular supermomentum complexes to analyze a closed gravitational system, **CS**, to analyze Trautman's radiative spacetimes, **TRS**, and to analyze Friedman universes, **FU**. We will compare the obtained results with our results obtained in past by performing analogical analyzis with help of the canonical superenergy tensors and the canonical angular supermomentum tensors.

We will see that the canonical superenergy and angular supermomentum complexes are, in some sense, complementary quantities to the canonical superenergy and angular momentum tensors. Namely, these complexes enable us global analysis of the solutions to the Einstein equations. But this analysis is coordinate-dependent. On the other hand, the canonical superenergy and angular supermomentum tensors are suitable to coordinate-independent, local analysis of such solutions.

II. APPLICATION TO A CLOSED GRAVITATIONAL SYSTEM

Henceforth we will use *geometrical units* in which $G = c = 1$.

By *closed system*, **CS**, we mean an isolated material system which admits global coordinates (t, x, y, z) in which metric components, g_{ik} , have the form

$$\begin{aligned} g_{ik} &= \eta_{ik} + h_{ik}, \\ h_{ik} &= O\left(\frac{1}{r}\right), \quad g_{ik,l} = O\left(\frac{1}{r^2}\right), \end{aligned} \quad (22)$$

where

$$r^2 = x^2 + y^2 + z^2, \quad (23)$$

and $\eta_{ik} = \text{diag}(1, -1, -1, -1)$.

The coordinates (t, x, y, z) are called *asymptotically flat Bondi-Sachs coordinates*. (For more detailed description of behaviour of the metric components for a **CS**, see [4]).

Integrating the continuity equations (14) over the spatial section $x^0 = t = \text{const}$, one gets (with help of the Stokes integral theorem, see, e.g., [1, 4])

$$\frac{d}{dt} \oint_{\partial x^0} \{ \delta^{ab} {}_F U_i^{[0\alpha]}{}_{,ab} n_\alpha d^2 \sigma \} = (-) \oint_{\partial x^0} \{ \delta^{ab} {}_F U_i^{[\alpha l]}{}_{,lab} n_\alpha d^2 \sigma \}. \quad (24)$$

Here, ∂x^0 , means 2-dimensional boundary of the spatial slice $x^0 = t = \text{const}$; n_α denotes spatial components of the unit normal \vec{n} to this boundary directed outside, and $d^2 \sigma = r^2 \sin \theta d\theta d\varphi$.

The equations (24) represent the four integral conservation laws for canonical superenergy ${}_K S_0$ and supermomentum ${}_K S_\alpha$:

$$\frac{d}{dt} {}_K S_i = (-) \oint_{\partial x^0} {}_K S_i{}^\alpha n_\alpha d^2 \sigma, \quad (25)$$

where

$${}_K S_i := \int_{x^0=\text{const}} (\delta^{ab} {}_F U_i^{[0\alpha]}{}_{,ab}){}_{,\alpha} d^3 v = \oint_{\partial x^0} \delta^{ab} {}_F U_i^{[0\alpha]}{}_{,ab} n_\alpha d^2 \sigma \quad (26)$$

and

$${}_K S_i{}^\alpha := \delta^{ab} {}_F U_i^{[\alpha l]}{}_{,lab}. \quad (27)$$

One can easily calculate (See, e.g., [4]) that for a closed system **CS**

$${}_F U_i^{[0\alpha]}{}_{,ab} n_\alpha = O\left(\frac{1}{r^4}\right), \quad (28)$$

$${}_F U_i^{[\alpha l]}{}_{,lab} = O\left(\frac{1}{r^4}\right). \quad (29)$$

Therefore, the *integral conservation laws* (24) [or (25)] *trivialize* to the form $0 = 0$ in the case because $d^2 \sigma = O(r^2)$, and *have no physical meaning*.

This is reasonable result because we have in the case four ordinary integral conservation laws of the global energy-momentum (See, e.g., [4]), and, in consequence, we needn't any additional integral conservation laws which would have been physically valid.

In a similar way the continuity equations (20) lead us finally to the following six integral equalities

$$\begin{aligned} \frac{d}{dt} \int_{x^0=\text{const}} {}_K S^{ij0} d^3 v &= (-) \int_{x^0=\text{const}} {}_K S^{ij\alpha}{}_{,\alpha} d^3 v \\ &= (-) \oint_{\partial x^0} {}_K S^{ij\alpha} n_\alpha d^2 \sigma, \end{aligned} \quad (30)$$

or

$$\frac{d}{dt}M^{ij} = (-) \oint_{\partial x^0} {}_K S^{ij\alpha} n_\alpha d^2\sigma. \quad (31)$$

In extended form

$$\begin{aligned} M^{ij} &= (-)M^{ji} := \int_{x^0=\text{const}} {}_K S^{ij0} d^3v = \int_{x^0=\text{const}} (\delta^{ab} M^{ij0})_{,ab} d^3v \\ &= \oint_{\partial x^0} (\delta^{ab} M^{[ij][0\alpha]}_{,ab}) n_\alpha d^2\sigma, \end{aligned} \quad (32)$$

and

$${}_K S^{ij\alpha} := \delta^{ab} M^{[ij][\alpha l]}_{,lab}. \quad (33)$$

The equations (31) could give us six integral conservation laws of the canonical angular supermomentum $M^{ij} = (-)M^{ji}$. But, one can calculate that for a **CS** (See, eg., [4])

$${}_F U^{j[0\alpha]}_{,b} n_\alpha = O(r^{-3}), \quad {}_F U^{j[0\alpha]}_{,ab} n_\alpha = O(r^{-4}). \quad (34)$$

Substituting these asymptotics into integrals in (31) we obtain

$$M^{ij} = (-)M^{ji} = 0, \quad \oint_{\partial x^0} \delta^{ab} M^{[ij][\alpha l]}_{,lab} n_\alpha d^2\sigma = 0, \quad (35)$$

i.e., we again obtain six trivial equalities $0 = 0$.

Like as it was in the case of the canonical superenergy and supermomentum the last result is very satisfactory because we have here already six integral conservation laws for angular momentum and we needn't any other nontrivial integral conservation laws.

III. APPLICATION TO TRAUTMAN'S RADIATIVE SPACETIMES

By Trautman's radiative spacetime, **TRS**, we mean vacuum solution to the Einstein equations admitting asymptotically flat coordinates (t, x, y, z) in which one has

$$\begin{aligned} g_{ik} &= \eta_{ik} + O(r^{-1}), \quad g_{ik,l} = I_{ik} k_l + O(r^{-2}), \quad I_{ik} = O(r^{-1}), \\ (I_{ik} - \frac{1}{2} \eta_{ik} \eta^{ab} I_{ab}) k^k &= O(r^{-2}), \\ g_{ik,lm} &= J_{ik} k_l k_m + O(r^{-2}), \quad J_{ik} = J_{ki} = O(r^{-1}), \\ \Gamma^a_{bc} &= \Gamma^a_{cb} = O(r^{-1}), \quad R_{iklm} = O(r^{-1}). \end{aligned} \quad (36)$$

k^i are components of a null vector directed to scri-plus $,S^+$, and $r = \sqrt{x^2 + y^2 + z^2}$ (For more details, see e.g., [4, 13, 14]).

Trautman's radiative spacetimes admit outgoing gravitational radiation.

One can calculate (See, e.g., [4]) that for, **TRS**, one has in the coordinates (t, x, y, z)

$$\begin{aligned} {}_F U_i^{[kl]}_{,l} &= O(r^{-2}), \quad {}_F U_i^{[0\alpha]} n_\alpha = O(r^{-2}), \\ {}_F U_i^{[0\alpha]}_{,\alpha} &= O(r^{-2}). \end{aligned} \quad (37)$$

From this it follows that

$$\begin{aligned} [(\delta^{ab} {}_F U_i^{[0\alpha]}_{,ab})_{,\alpha}] &= O(r^{-2}), \quad [(\delta^{ab} {}_F U_i^{[0\alpha]}_{,ab}) n_\alpha] = O(r^{-2}), \\ {}_F U_i^{[0\alpha]}_{,b\alpha} &= O(r^{-2}). \end{aligned} \quad (38)$$

Using the above asymptotics one obtains that the integrals (26) on S_i are convergent and different from zero but they depend on time.

The integrals on the right hand of (24) (or (25)) are also convergent.

So, in the case of **TRS** the equations (24) (or (25)) do not trivialize to the form $0 = 0$ but they give us the laws of the temporal change of the integral canonical superenergetic quantities S_i .

Similar situation we have in the case for the canonical energetic quantities (See, e.g., [4, 14]).

Therefore, in **TRS**, where we have convergent but depended on time integral energetic quantities, like energy and momentum, the integral superenergetic quantities become nontrivial and can be physically valid.

Concerning components of the angular supermomentum complex ${}_K S^{ijk} = (-) {}_K S^{jik}$, one can easily see from the extended formulas (30)-(33) and from the asymptotics (38) that the integrals on $M^{ij} = (-) M^{ji}$ are *divergent* in **TRS** [$(\delta^{ab} M^{[ij][0\alpha]}{}_{,ab}) n_\alpha = O(r^{-1})$ in the case].

Thus, we have in the case the same situation as in the case of the ordinary angular momentum.

IV. FORMAL APPLICATION OF THE CANONICAL SUPERENERGY COMPLEX ${}_K S_i{}^k$ TO ANALYZE FRIEDMAN UNIVERSES

Here we confine to canonical superenergy complex only.

If one formally uses the canonical superenergy complex

$${}_K S_i{}^k = (\delta^{ab} {}_K K_i{}^k)_{,ab} = {}_F U_i{}^{[kl]}{}_{,lab} \delta^{ab} \quad (39)$$

to analyze Friedman universess in “Carthesian” coordinates (t, x, y, z) in which Friedman line element reads

$$ds^2 = dt^2 - \frac{R^2(t)(dx^2 + dy^2 + dz^2)}{[1 + \frac{k}{4}(x^2 + y^2 + z^2)]^2}, \quad (40)$$

where the *curvature index* $k = 0, \pm 1$, and $R = R(t)$ is the so-called *scale factor*, then, after some calculations, one can see that this complex is better to this aim than the canonical energy-momentum complex ${}_E K_i{}^k = {}_F U_i{}^{[kl]}{}_{,l}$ e.g., it better suits to singularity analysis in Friedman universes than the complex ${}_E K_i{}^k$ [5, 6]. (See also below).

In the flat case $k = 0$ all “densities” of the canonical superenergetic quantities *trivially vanish* being multiplied by $k = 0$. In consequence, the formally calculated integral canonical superenergy quantities also trivially vanish in the case.

It is interesting that the same result we have for canonical energy-momentum for flat Friedman universes (See, e.g., [5]).

In the cases $k = \pm 1$ the “densities” of the canonical superenergetic quantities are different from zero and all go to $(-)\infty$ when $R(t) \rightarrow 0^+$.

Therefore, these “densities” can be used to analysis of the initial singularities in these cases.

We would like to remark that the canonical energetic quantities are not relevant to this aim because all their “densities” go to zero when the scale factor goes to zero (See, e.g., [5]).

The formally calculated integral canonical superenergetic quantities for Friedman universes having $k = \pm 1$ read

$$S_0 = \int_{t=\text{const}} {}_K S_0{}^0 d^3v = \begin{cases} \frac{49\chi\pi^2 R}{8} > 0, & \text{if } k=1; \\ (-)\infty, & \text{if } k=(-)1 \end{cases}, \quad (41)$$

and

$$S_\beta := \int_{t=\text{const}} {}_K S_\beta{}^0 dx dy dz = 0. \quad (42)$$

Here S_β are components of integral linear supermomentum and $\chi = \frac{1}{16\pi}$.

The above results are very similar to the results obtained in past for integral energetic quantities (except $E = P_0 = 0$ if $k = 1$) [5].

We must emphasize that there exist natural objections against integral quantities for Friedman universes because these universes are not asymptotically flat: they are conformally flat only. In consequence, the integral quantities of the Friedman universes are *not measurable*. Therefore, considerations of these quantities for Friedman universes can have some mathematical meaning only.

Finishing this Section we would like to remark that from both, geometrical and physical points of view, the using of the our canonical superenergy tensors ${}_g S_i^k$, ${}_m S_i^k$, gravitation and matter, to analyze Friedman universe is much more reasonable than the using to this goal the canonical superenergy complex. For example, the canonical superenergy densities, ϵ_s , are positive definite scalars for all **FU** and they are singular when $R(t) \rightarrow 0^+$.

V. CANONICAL SUPERENERGY AND ANGULAR SUPERMOMENTUM COMPLEXES VERSUS CANONICAL SUPERENERGY AND ANGULAR SUPERMOMENTUM TENSORS

In past we have introduced the canonical superenergy and angular supermomentum tensors, matter and gravitation and total, by using special averaging of the differences of the canonical energy-momentum in Riemann normal coordinates, **RNC(P)**, [5] (See also Appendix). **P** is the beginning of these coordinates.

These tensors, constructed pointwise, were very suitable to local, coordinate independent analysis of the gravitational field, and also to analyze matter field. Moreover, the canonical superenergy tensor for gravitational field, ${}_g S_i^k(P; v^l)$, gave us some substitute of the non-existing gravitational energy-momentum tensor.

The constructive definitions of the canonical superenergy tensors, ${}_g S_i^k$, ${}_g S^{ikl}$, ${}_m S_i^k$, ${}_m S^{ikl}$, immediately led us at first to 4-dimensional ‘‘Laplacians’’ $\delta^{ab} \partial_a \partial_b$ at point **P** of the averaged fields. Then, after expressing these ‘‘Laplacians’’ covariantly in terms of curvature tensor and its comitants, in terms of 4-velocity of a fiducial observer **O** which is at rest at point **P** and in terms of covariant derivatives of matter tensor, we have obtained the our superenergy and angular supermomentum tensors ${}_g S_i^k$, ${}_m S_i^k$, ${}_g S^{ikl}$, ${}_m S^{ikl}$.

About four years ago we have observed that these tensors can be also obtained more easily from the canonical superenergy and angular supermomentum complexes ${}_K S_i^k$, ${}_K S^{ikl} = (-)_K S^{kil}$.

With this aim it is sufficient to take the components of these complexes at point **P**= beginning of the **RNC(P)** and express them covariantly by using special properties of the **RNC(P)** [7, 8].

In this paper we have defined the canonical superenergy and angular supermomentum complexes ${}_K S_i^k$, ${}_K S^{ikl} = (-)_K S^{kil}$, and applied them to analyze of a **CS**, **TRS**, and to analyze (superenergetic only) Friedman universes, **FU**.

In the case of a **CS** the integral canonical superenergetic quantities satisfy four trivial conservation laws $0 = 0$, and in the case **TRS** they behave very similar to the integral canonical energetic quantities.

On the other hand, it seems that the canonical superenergy complex ${}_K S_i^k$ gives better tool to analysis Friedman universes in ‘‘Cartesian’’ coordinates (t, x, y, z) than the canonical energy-momentum complex ${}_E K_i^k$ [16].

In our previous papers [5, 6] we have applied the canonical superenergy tensors, matter and gravitation, ${}_m S_i^k$, ${}_g S_i^k$, to local, and in some special cases, also to global analysis of the very known solutions to the Einstein equations.

The obtained results were interesting [5, 6] (See also Appendix). In general one can say that the canonical superenergy and angular supermomentum tensors give better tool to local analysis of these solutions. Moreover, the canonical angular supermomentum tensors, as being independent of any radius vector, lead to better convergence of the suitable integrals in **TRS**.

Recently we have proposed [5] to use the total canonical superenergy density $\epsilon_s := ({}_g S_i^k + {}_m S_i^k) v^i v_k$ to study gravitational stability of the solutions to the Einstein equations.

Comparing the two approaches to superenergy:

1. Canonical superenergy tensors ${}_g S_i^k$, ${}_m S_i^k$; $S_i^k := {}_g S_i^k + {}_m S_i^k$,
2. Canonical superenergy complex ${}_K S_i^k$, gravity and matter,

one can conclude that the canonical superenergy tensors give better tool to local (but in some cases also to global) analysis of the solutions to the Einstein equations than the canonical superenergy complex. It is mainly because they are tensors.

The canonical superenergy tensors, as a result of some averaging, do not satisfy in general any conservation laws, local or global. It is a defect of these tensors. But they refer to real gravitational field only and it is a positive property.

One can look on these tensors as on some kind of quasilocal quantities which are not conserved.

Contrary, the canonical superenergy complex ${}_K S_i^k$ satisfies conservation laws but it is coordinate dependent, non-tensorial quantity and, therefore, it can be reasonably use in special coordinates only, e.g., to global analysis of an asymptotically flat spacetime in an asymptotically flat coordinates.

The analogical properties possesses the canonical energy-momentum complex ${}_E K_i^k$.

One can see on the canonical superenergy complex ${}_K S_i^k$ as on a conserved quasilocal quantity.

The complex ${}_K S_i^k$, like as the canonical energy-momentum complex ${}_E K_i^k$, is a *mixture of real and fictive (=inertial) gravitational fields*.

Concerning canonical angular supermomentum complex ${}_K S^{ikl} = (-)_K S^{kil}$ one can say that it satisfies continuity equations (30)-(33), and, in the case of a **CS** it leads to six, trivial, integral conservation laws $0 = 0$. In **TRS** this complex behaves very alike to the canonical angular momentum complex $M^{ijk} = (-)M^{jik}$ leading to divergent integrals.

The canonical angular supermomentum tensors gravitation and matter, ${}_g S^{ikl}$, ${}_m S^{ikl}$, and total $S^{ikl} := {}_g S^{ikl} + {}_m S^{ikl}$, introduced in our previous papers [5, 6] (See also Appendix) give better tool to analyze gravitational and matter fields than the canonical angular supermomentum complex ${}_K S^{ikl} = (-)_K S^{kil}$ despite that they do not satisfy any continuity equations. For example, they are tensors and lead to better convergence integrals in **TRS** owing the fact that they *do not depend on components of any radius-vector* (See, e.g., [5, 6] and Appendix).

Similarly as it was in the case canonical superenergy, one can interpret the complex ${}_K S^{ikl}$ as a conserved quasilocal quantity, and the tensors ${}_g S^{ikl}$, ${}_m S^{ikl}$, S^{ikl} as non-conserved quasilocal quantities.

One can consider differences

$$\begin{aligned} R_i^K &:= {}_K S_i^k - S_i^k, \\ R^{ikl} &:= {}_K S^{ikl} - S^{ikl}, \end{aligned}$$

and connect them with *inertial forces*, i.e., with fictive gravitational field.

Here

$$S_i^k := {}_g S_i^k + {}_m S_i^k, \quad S^{ikl} := {}_g S^{ikl} + {}_m S^{ikl} \quad (43)$$

mean components of the total superenergy and total angular supermomentum tensors matter and gravitation respectively.

VI. APPENDIX

Here we remind constructive definition of the superenergy tensor S_a^b and analogical definition of the angular supermomentum tensor S^{ikl} applicable to gravitational field and to any matter field. The definitions use *locally Minkowskian structure* of the spacetime and, therefore, they fail in a spacetime with torsion, e.g., in Riemann-Cartan spacetime.

Let us start with the superenergy tensor.

In the normal Riemann coordinates **NRC(P)**, we define (pointwise)

$$S_{(a)}^{(b)}(P) = S_a^b := (-) \lim_{\Omega \rightarrow P} \frac{\int_{\Omega} \left[T_{(a)}^{(b)}(y) - T_{(a)}^{(b)}(P) \right] d\Omega}{1/2 \int_{\Omega} \sigma(P; y) d\Omega}, \quad (44)$$

where

$$\begin{aligned} T_{(a)}^{(b)}(y) &:= T_i^k(y) e_{(a)}^i(y) e_k^{(b)}(y), \\ T_{(a)}^{(b)}(P) &:= T_i^k(P) e_{(a)}^i(P) e_k^{(b)}(P) = T_a^b(P) \end{aligned} \quad (45)$$

are *physical or tetrad components* of the pseudotensor or tensor field which describes an energy-momentum distribution, and $\{y^i\}$ are normal coordinates. $e_{(a)}^i(y), e_k^{(b)}(y)$ denote an orthonormal tetrad $e_{(a)}^i(P) = \delta_a^i$ and its dual $e_k^{(a)}(P) = \delta_k^a$, paralelly propagated along geodesics through **P** (**P** is the origin of the **NRC(P)**).

We have

$$e_{(a)}^i(y) e_i^{(b)}(y) = \delta_a^b. \quad (46)$$

For a sufficiently small 4-dimensional domain Ω which surrounds **P** we require

$$\int_{\Omega} y^i d\Omega = 0, \quad \int_{\Omega} y^i y^k d\Omega = \delta^{ik} M, \quad (47)$$

where

$$M = \int_{\Omega} (y^0)^2 d\Omega = \int_{\Omega} (y^1)^2 d\Omega = \int_{\Omega} (y^2)^2 d\Omega = \int_{\Omega} (y^3)^2 d\Omega, \quad (48)$$

is a common value of the moments of inertia of the domain Ω with respect to the subspaces $y^i = 0$, ($i = 0, 1, 2, 3$).
As Ω we can take, e.g., a sufficiently small analytic ball centered at \mathbf{P} :

$$(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 \leq R^2, \quad (49)$$

which for an auxiliary positive-definite metric

$$h^{ik} := 2v^i v^k - g^{ik}, \quad (50)$$

can be written in the form

$$h_{ik} y^i y^k \leq R^2. \quad (51)$$

A fiducial observer \mathbf{O} is at rest at the beginning \mathbf{P} of the Riemann normal coordinates ,**NRC**(\mathbf{P}), and its four-velocity is $v^i = * \delta_o^i$. $= *$ means that equation is valid only in special coordinates. $\sigma(P; y)$ denotes the two-point *world function* introduced by J.L. Synge [9]:

$$\sigma(P; y) = \frac{1}{2} (y^0{}^2 - y^1{}^2 - y^2{}^2 - y^3{}^2). \quad (52)$$

The world function $\sigma(P; y)$ can be defined covariantly by the *eikonal-like equation* [9]

$$g^{ik} \sigma_{,i} \sigma_{,k} = 2\sigma, \quad \sigma_{,i} := \partial_i \sigma, \quad (53)$$

together with requirements

$$\sigma(P; P) = 0, \quad \partial_i \sigma(P; P) = 0. \quad (54)$$

The ball Ω can also be given by the inequality

$$h^{ik} \sigma_{,i} \sigma_{,k} \leq R^2. \quad (55)$$

Tetrad components and normal components are equal at \mathbf{P} , so, we will write the components of any quantity attached to \mathbf{P} without tetrad brackets, e.g., we will write $S_a{}^b(P)$ instead of $S_{(a)}{}^{(b)}(P)$ and so on.

If $T_i{}^k(y)$ are the components of an energy-momentum tensor of matter, then we get from (44)

$${}_m S_a{}^b(P; v^l) = (2\hat{v}^l \hat{v}^m - \hat{g}^{lm}) \nabla_l \nabla_m \hat{T}_a{}^b = \hat{h}^{lm} \nabla_l \nabla_m \hat{T}_a{}^b. \quad (56)$$

Hat over a quantity denotes its value at \mathbf{P} , and ∇ means covariant derivative.

Tensor ${}_m S_a{}^b(P; v^l)$ is called *the canonical superenergy tensor for matter*.

For the gravitational field, substitution of the canonical Einstein energy-momentum pseudotensor as $T_i{}^k$ into (44) gives

$${}_g S_a{}^b(P; v^l) = \hat{h}^{lm} \hat{W}_a{}^b{}_{lm}, \quad (57)$$

where

$$\begin{aligned} W_a{}^b{}_{lm} = & \frac{2\alpha}{9} [B_{alm}^b + P_{alm}^b \\ & - \frac{1}{2} \delta_a^b R^{ijk}{}_m (R_{ijkl} + R_{ikjl}) + 2\delta_a^b \beta^2 E_{(l|g} E_{|m)}^g \\ & - 3\beta^2 E_{a(l} E_{|m)}^b + 2\beta R_{(a|g|l)}^b E_{m)}^g]. \end{aligned} \quad (58)$$

Here $\alpha = \frac{c^4}{16\pi G} = \frac{1}{2\beta}$ [17], and

$$E_i{}^k := T_i{}^k - \frac{1}{2} \delta_i^k T \quad (59)$$

is the *modified energy-momentum tensor* of matter [18].

On the other hand

$$B_{alm}^b := 2R^{bik}{}_{(l} R_{aik|m)} - \frac{1}{2} \delta_a^b R^{ijk}{}_l R_{ijkm} \quad (60)$$

are components of the *Bel-Robinson tensor* (**BRT**), while

$$P_{alm}^b := 2R^{bik}_{(l|R_{aki|m})} - \frac{1}{2}\delta_a^b R^{jik}_{(l} R_{jkim)} \quad (61)$$

is the Bel-Robinson tensor with “transposed” indices (ik) .

In vacuum ${}_g S_a{}^b(P; v^l)$ takes the simpler form

$${}_g S_a{}^b(P; v^l) = \frac{8\alpha}{9} \hat{h}^{lm} (\hat{C}^{bik}_{(l} \hat{C}_{aik|m})} - \frac{1}{2} \delta_a^b \hat{C}^{i(kp)}_{(l} \hat{C}_{ikp|m})}). \quad (62)$$

Here C_{blm}^a denote components of the *Weyl tensor*.

The canonical angular supermomentum tensors, analogous to the case of the canonical superenergy tensors, can be considered as *some substitutes* for the angular momentum tensors of matter and gravitation which do not exist in **GR**.

The constructive definition of these tensors is the following. In analogy to the definition (44) of the canonical superenergy tensors we define in **RNC(P)**

$$S^{(a)(b)(c)} = S^{abc}(P) := - \lim_{\Omega \rightarrow P} \frac{\int_{\Omega} [M^{(a)(b)(c)}(y) - M^{(a)(b)(c)}(P)] d\Omega}{1/2 \int_{\Omega} \sigma(P; y) d\Omega}, \quad (63)$$

where

$$M^{(a)(b)(c)}(y) := M^{ikl}(y) e_i^{(a)}(y) e_k^{(b)}(y) e_l^{(c)}(y), \quad (64)$$

$$M^{(a)(b)(c)}(P) := M^{ikl}(P) e_i^{(a)}(P) e_k^{(b)}(P) e_l^{(c)}(P) = M^{ikl} \delta_i^a \delta_k^b \delta_l^c = M^{abc}(P), \quad (65)$$

are the physical (or tetrad) components of the field $M^{ikl}(y) = -M^{kil}(y)$ which describes angular momentum densities.

$e^i_{(a)}(y)$, $e_k^{(b)}(y)$ mean orthonormal base such that $e^i_{(a)}(P) = \delta_a^i$ and its dual *parallelly propagated along geodesics through P*. Ω is an already defined sufficiently small analytic ball with centre **P**.

At the point **P** we have equality of the tetrad and normal components. We use this fact and omit tetrad brackets for indices of any quantity attached at the point **P**; for example, we write $S^{abc}(P)$ instead of $S^{(a)(b)(c)}(P)$ and so on.

For matter, as ${}_m M^{ikl}(y) = (-)_m M^{kil}(y)$ we take

$${}_m M^{ikl}(y) = \sqrt{|g|} (y^i T^{kl} - y^k T^{il}), \quad (66)$$

where $T^{ik} = T^{ki}$ are the components of a symmetric energy-momentum tensor of matter and y^i denote, as usual, Riemann normal coordinates.

The expression (66) gives us *total angular momentum densities*, orbital and spinorial ones because the dynamical energy-momentum tensor for matter $T^{ik} = T^{ki}$ is obtained from the canonical one by means of the *Belinfante-Rosenfeld* symmetrization procedure, and, therefore, includes material spin [11].

Note that the normal coordinates y^i form the components of the local *radius vector* \vec{y} with respect to the origin **P**. [19]

In consequence, the components of the ${}_m M^{ikl}(y)$ form a local tensor density.

For the gravitational field we favourize and take the expression proposed by Bergmann and Thomson [11] as the gravitational angular momentum pseudotensor

$${}_g M^{ikl}(y) = {}_F U^{i[kl]}(y) - {}_F U^{k[il]}(y) + \sqrt{|g|} (y^i {}_{BT} t^{kl} - y^k {}_{BT} t^{il}), \quad (67)$$

where

$${}_F U^{i[kl]} := g^{im} {}_F U_m^{[kl]} \quad (68)$$

are the Freud superpotentials with the first index raised and

$${}_{BT} t^{kl} := g^{ki} {}_{Et}^l_i + \frac{g^{mk}}{\sqrt{|g|}} {}_F U_m^{[lp]} \quad (69)$$

is *Bergmann-Thomson* gravitational energy-momentum pseudotensor.

One can easily see that the expressions (66)-(67) are exactly the material $({}_m M^{ikl})$ and the gravitational $({}_g M^{ikl})$ parts of the canonical angular momentum complex (15).

The expressions (66)-(67) are most closely related to the Einstein canonical energy-momentum complex ${}_E K_i^k$. That is why we have applied them here and called them *canonical*.

One can interpret the Bergmann-Thomson gravitational angular pseudotensor as the sum of the *spinorial part*

$$S^{ikl} := {}_F U^{i[kl]} - {}_F U^{k[il]} \quad (70)$$

and the *orbital part*

$$O^{ikl} := \sqrt{|g|} (y^i {}_B T^{kl} - y^k {}_B T^{il}) \quad (71)$$

of the gravitational angular momentum “densities”.

Substitution of (66) and (67) (expanded up to third order) into (63) gives the *canonical angular supermomentum tensors* for matter and gravitation respectively

$${}_m S^{abc}(P; v^l) = 2[(2v^a v^p - g^{ap}) \nabla_p T^{bc} - (2v^b v^p - g^{bp}) \nabla_p T^{ac}], \quad (72)$$

$$\begin{aligned} {}_g S^{abc}(P; v^l) = & \alpha(2v^p v^t - g^{pt}) [(g^{ac} g^{br} - g^{bc} g^{ar}) \nabla_{(t} R_{p)r)} \\ & + 2g^{ar} \nabla_{(t} R_{p \quad r)}^{(b \quad c)} - 2g^{br} \nabla_{(t} R_{p \quad r)}^{(a \quad c)} \\ & + \frac{2}{3} g^{bc} (\nabla_r R_{(t \quad p)}^r{}^a - \nabla_{(p} R_{t)}^a{}^r) - \frac{2}{3} g^{ac} (\nabla_r R_{(t \quad p)}^r{}^b - \nabla_{(p} R_{t)}^b{}^r)]. \end{aligned} \quad (73)$$

Both these tensors are antisymmetric in the first two indices.

In vacuum, the gravitational canonical angular supermomentum tensor (73) simplifies to

$${}_g S^{abc}(P; v^l) = 2\alpha(2v^p v^t - g^{pt}) [g^{ar} \nabla_{(p} R_{t \quad r)}^{(b \quad c)} - g^{br} \nabla_{(p} R_{t \quad r)}^{(a \quad c)}]. \quad (74)$$

Note that the orbital part

$$O^{ikl} = \sqrt{|g|} (y^i {}_B T^{kl} - y^k {}_B T^{il}) \quad (75)$$

gives no contribution to ${}_g S^{abc}(P; v^l)$. Only the *spinorial part*

$$S^{ikl} = {}_F U^{i[kl]} - {}_F U^{k[il]} \quad (76)$$

contributes.

Also, the canonical angular supermomentum tensors ${}_g S^{abc}(P; v^l)$ and ${}_m S^{abc}(P; v^l)$ needn't any radius vector too their own existing.[20]

Some final remarks:

1. In vacuum the quadratic form ${}_g S_a^b v^a v_b$, where $v^a v_a = 1$, is *positive-definite*. This form gives the gravitational *superenergy density* ϵ_g for a fiducial observer \mathbf{O} .
2. In general, the canonical superenergy and angular supermomentum tensors are uniquely determined only along the world line of an observer \mathbf{O} . But in special cases, e.g., in Schwarzschild spacetime or in Friedmann universes, when there exists a physically and geometrically distinguished four-velocity field, $v^i(x)$, one can introduce, in an unique way, unambiguous fields ${}_g S_i^k(x; v^l)$ and ${}_m S_i^k(x; v^l)$.

If we assume that the spacetime is *globally hyperbolic*, then there also exists a distinguished, global timelike vector field $\vec{v} : g(v, v) = 1$. One can use this vector field to global construction of the canonical superenergy and angular supermomentum tensor fields.

3. It can be shown that the superenergy densities ϵ_g, ϵ_m , which have dimension $\frac{\text{Joul}}{\text{metre}^5}$, exactly correspond to the Appel's *energy of acceleration* $\frac{1}{2} \vec{a} \vec{a}$. The Appel's energy of acceleration plays fundamental role in Appel's approach to classical mechanics.[10] We have already told about that in main text.
4. Recently we have noticed that the total superenergy density is positive-definite or null for known gravitationally stable solutions to the Einstein equations and negative-definite for gravitationally unstable solutions. We have used this fact to study gravitational stability, i.e., stability with respect small metric perturbations, of many very known solutions to the Einstein equations. [5]
5. By using canonical gravitational superenergy and angular supermomentum tensors one can prove that the exact gravitational waves carry energy- momentum and angular momentum. [5]

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 - [15] Complex because the components of ${}_K S_i^k$, like the components ${}_E K_i^k$, do not form any tensor.
 - [16] But much worse tool than the canonical superenergy tensors ${}_g S_i^k$, ${}_m S_i^k$
 - [17] In geometrized units $\alpha = \frac{1}{16\pi}$
 - [18] In terms of E_i^k Einstein equations read $R_i^k = \beta E_i^k$.
 - [19] A global radius vector does not exist in **GR**.
 - [20] This is very good property because any global radius vector does not exist in **GR**.